Simple Sticky-Wage New Keynesian Model

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1 Overview

1.1 Objective

The motivation for this note is to address the following three concerns:

1. Deriving the New Keynesian model is often algebraically painful, but it need not be so.
2. Sticky-wage New Keynesian models in continuous time are scarce, so it is difficult to find a derivation.
3. Regardless of discrete or continuous time setting, most sticky-wage models endow the household side of the economy with market power via wage-setting. Instead, the following model endows firms with monopsonistic wage-setting power.

As a bonus fourth point, I will not need to employ any linearization to arrive at the final equilibrium system; everything comes from agents optimizing and imposing representativeness and markets clearing. My approach is most similar to Ben Moll's lecture notes, through I slightly generalize and focus on wage rigidity instead of price rigidity.

To achieve these goals, I explain the assumptions and solve the model up to finding the equilibrium conditions. I go into detail showing the mathematical steps taken, first finding the optimality conditions for households and firms, and then imposing equilibrium. After finding the equilibrium equations, I show the outcomes from four different shock experiments, solving for the time paths numerically, and explaining the outcomes.

1.2 Model

I now describe all the pieces of the model, and in the following sections break down the relevant agent problems to find the equilibrium conditions.

Time is continuous and we focus on perfect foresight equilibria. A representative household chooses how much to work, consume, and correspondingly save. At each instant, they face a wage from each firm $\omega \in [0, 1]$ and a goods price. They view jobs at different firms as differentiated, and have a love of variety over jobs, giving firms monopsonistic power in the labor market. Households also face a nominal interest rate which they earn by saving in bonds. Firms choose a path for wages, but face a convex cost in the speed of wage growth/decay, and seek to maximize the present discounted value of profits.

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Although none of the mechanics or functional forms have been detailed, it is instructive to explain the equilibrium concept now, so that we have an idea of where we are headed in solving each subproblem. The equilibrium is a set of paths for prices, wages, consumption, and labor, such that

(i) Given all prices, households choose the consumption and labor paths, which imply a bond (demand) path.

(ii) Given all prices and aggregates, individual firm wage paths match the aggregate wage path.

(iii) Goods market clears.

(iv) Labor markets clear.

(v) Bond market clears ($B = 0$ for all time).

2 Household

The household problem may be broken into an "inner" problem of finding the optimal bundle of labor, given the wage schedule, and an "outer" problem of finding optimal consumption, savings, and labor supply, given the relevant prices implied by the solution of the inner problem. We will work from the inside out, first solving the inner problem, which will yield a wage index that can be used in solving the outer problem.

2.1 Labor Bundle

Households (dis)value labor flows according to the constant elasticity of substitution aggregator below. The inner problem is then to solve "Given a required bundle N and wage schedule $w(\omega)$, what is the maximal earnings way to only work N?". A unit continuum of firms exist, indexed by $\omega \in [0,1]$, and each has a job, so households solve the following problem, for a given level of labor bundle N[1]

$$\max_{n(\omega)} \int_0^1 w(\omega) n(\omega) \mathrm{d}\omega \tag{1}$$

$$N = \left(\int_0^1 n(\omega)^{\frac{\eta+1}{\eta}} \mathrm{d}\omega \right)^{\frac{\eta}{\eta+1}} \tag{2}$$

I solve this problem via pointwise optimization. First, set up the Langrangian with multiplier λ on the constraint.

$$\mathcal{L} = \int_0^1 w(\omega) n(\omega) \mathrm{d}\omega + \lambda \left[N - \left(\int_0^1 n(\omega)^{\frac{\eta+1}{\eta}} \mathrm{d}\omega \right)^{\frac{\eta}{\eta+1}} \right] \tag{3}$$

The first-order conditions are

[1] Why is the power term $\frac{\eta+1}{\eta}$ instead of $\frac{\eta-1}{\eta}$, as we usually see for CES preferences over goods? The standard $-$ form generates isoelastic demand over varieties, where the $+$ form here generates isoelastic supply. Note also that the goal of agents is to maximize earnings, given bundle N, whereas with CES over goods, the goal of agents is to minimize expenditures, given bundle C.

$$w(\omega) = \lambda \left(\frac{n(\omega)}{N}\right)^{\frac{1}{\eta}} \tag{4}$$

Then I can use the constraint to find λ in terms of total labor earnings

$$\frac{w(\omega)n(\omega)}{\lambda} = n(\omega)^{\frac{\eta+1}{\eta}} N^{-\frac{1}{\eta}} \tag{5}$$

$$\Rightarrow \frac{\int_0^1 w(\omega)n(\omega)\mathrm{d}\omega}{\lambda} = \left(\int_0^1 n(\omega)^{\frac{\eta+1}{\eta}} \mathrm{d}\omega\right) N^{-\frac{1}{\eta}} \tag{6}$$

$$= N \tag{7}$$

So λ is the unit earnings of the bundle of total labor, and this information allows us to solve for λ explicitly in terms of wages.

$$\lambda N = \int_0^1 w(\omega)n(\omega)\mathrm{d}\omega \tag{8}$$

$$= \int_0^1 w(\omega)\lambda^{-\eta} w(\omega)^{\eta} N \mathrm{d}\omega \tag{9}$$

$$\Rightarrow \lambda = \left(\int_0^1 w(\omega)^{\eta+1}\mathrm{d}\omega\right)^{\frac{1}{\eta+1}} \tag{10}$$

This motivates calling λ the aggregate wage index, $W \equiv \lambda$, so that $WN = \int_0^1 w(\omega)n(\omega)\mathrm{d}\omega$, as we might have hoped. Then the supply system fully in terms of wages and aggregate labor is

$$n(\omega) = \left(\frac{w(\omega)}{W}\right)^{\eta} N \tag{11}$$

$$W = \left(\int_0^1 w(\omega)^{1+\eta}\mathrm{d}\omega\right)^{\frac{1}{1+\eta}} \tag{12}$$

2.2 Consumption Bundle

Households view goods from different firms as perfectly substitutable, and thus at every moment in time the goods price index is simply the minimal price offered, and the goods bundle is the amount of consumption that can be bought at that minimal price. This may be thought of as solving the following problem in the limit as $\epsilon \to \infty$, and in fact finite ϵ do not materially affect the results below, other than changing the flexible outcome wedge between wages and the marginal revenue product of labor (or simple the wage markdown) from $\frac{\eta}{\eta+1}$ to $\frac{\eta}{\eta+1}\frac{\epsilon-1}{\epsilon}$.

$$\min_{c(\omega)} \int_0^1 p(\omega)c(\omega)\mathrm{d}\omega \tag{13}$$

$$C = \left(\int_0^1 c(\omega)^{\frac{\epsilon-1}{\epsilon}}\mathrm{d}\omega\right)^{\frac{\epsilon}{\epsilon-1}} \tag{14}$$

Following similar steps as above for the labor bundle, we would find the relevant price index as the following.

$$P = \left(\int_0^1 p(\omega)^{1-\epsilon} \mathrm{d}\omega \right)^{\frac{1}{1-\epsilon}} \tag{15}$$

The limit $\epsilon \to \infty$ indeed delivers

$$P = \min_{\omega} p(\omega) \tag{16}$$

In equilibrium, all firms will offer the same price, and thus the household will be indifferent between all bundles of consumption, but the only bundle that will clear all the goods markets is if the household splits its consumption evenly among all firms.

2.3 Consumption, Savings, and Labor

Given the *distribution* of hours a household works, and the implied relevant wage index W from the inner problem solution, households must choose the *scale* of labor N, as well as consumption C, and thus savings B, at every moment in time t. They discount the future at rate ρ, earn nominal interest i on any bonds B (or debt $-B$) they hold, earn wages W, and receive dividends D from firm profits and adjustment costs[2]. Their instantaneous flow consumption utility is $U(\cdot)$ and flow labor disutilty is $\tilde{U}$. They start with bonds B_0 (which will be zero in equilibrium below). Formally, they solve

$$V(B_0, t_0) \equiv \max_{C,N} \int_{t_0}^{\infty} e^{-\rho(t-t_0)} \left[U(C(t)) - \tilde{U}(N(t)) \right] \mathrm{d}t \tag{17}$$

$$\dot{B}(t) = i(t)B(t) + W(t)N(t) - P(t)C(t) + D(t) \tag{18}$$

$$B(0) = B_0 \tag{19}$$

Standard Hamiltonian methods may be used to solve this optimal control problem, or a recursive formulation may be used. I exposit the latter approach. The Hamilton-Jacobi-Bellman equation may be derived by totally differentiating the value function V with respect to t_0 (note that the following expression assumes the maximal C^* and N^* are being chosen)

$$V_B \dot{B} + V_t = -U(C^*) + \tilde{U}(N^*) + \rho V \tag{20}$$

Rearranging gives the standard form, where I now reinclude the maximization problem that must be solved for (20) to hold.

$$\rho V(B,t) - V_t(B,t) = \max_{C,N} \left[U(C) - \tilde{U}(N) \right] + V_B \left[i(t)B(t) + W(t)N - P(t)C + D(t) \right] \tag{21}$$

The first-order conditions for optimal consumption and labor are

[2] Payments for firm wage adjustment costs go into D so that nominal rigidities only cause misallocation, but do not directly destroy goods.

$$U'(C) = P(t)V_B \tag{22}$$
$$\tilde{U}'(N) = W(t)V_B \tag{23}$$

Thus, we find the consumption-labor tradeoff

$$\frac{\tilde{U}'(N)}{U'(C)} = \frac{W(t)}{P(t)} \tag{24}$$

To derive the consumption Euler equation, I make use of the envelope theorem and differentiate V with respect to B

$$\rho V_B - V_{tB} = V_{BB}\dot{B} + V_B i(t) \tag{25}$$

I also differentiate the consumption first-order condition with respect to time

$$U''(C)\dot{C} = \dot{P}V_B + P(t)V_{BB}\dot{B} + P(t)V_{Bt} \tag{26}$$

I combine these expressions, as well as the original consumption first-order condition, to find

$$\rho\frac{U'(C(t))}{P(t)} = \frac{U''(C)\dot{C}}{P} - \frac{\dot{P}}{P}\frac{U'(C)}{P} + \frac{U'(C(t))}{P(t)}i(t) \tag{27}$$

Rearranging gives the Euler equation

$$\frac{\dot{C}}{C} = -\frac{U'(C)}{U''(C)C}(i(t) - \pi(t) - \rho) \tag{28}$$

where $\pi(t) \equiv \frac{\dot{P}(t)}{P(t)}$ is goods price inflation. To find the labor Euler equation, either follow analogous steps as above, but using the labor conditions, or time-differentiate the consumption labor tradeoff (easiest to log both sides first) and combine with the consumption Euler equation. I will do the latter.

$$\frac{\tilde{U}''(N)}{\tilde{U}'(N)}\dot{N} - \frac{U''(C)}{U'(C)}\dot{C} = \frac{\dot{W(t)}}{W} - \frac{\dot{P}(t)}{P(t)} \tag{29}$$

Therefore

$$\begin{aligned}
\frac{\dot{N}}{N} &= \frac{\tilde{U}'(N)}{\tilde{U}''(N)N}\left(\frac{\dot{W}(t)}{W} + \frac{U''(C)}{U'(C)}\dot{C}\right) && (30)\\
&= \frac{\tilde{U}'(N)}{\tilde{U}''(N)N}\left(\frac{\dot{W}(t)}{W} - \frac{\dot{P}(t)}{P(t)} - [i(t) - \pi(t) - \rho]\right) && (31)\\
&= -\frac{\tilde{U}'(N)}{\tilde{U}''(N)N}\left(i(t) - \psi(t) - \rho\right) && (32)
\end{aligned}$$

where $\psi(t) \equiv \frac{\dot{W}(t)}{W(t)}$ is wage inflation. The consumption-labor equation (24) and Euler equations (28) and (32) fully characterize household behavior under deterministic paths for prices. For simplicity, I will take $U(C)$ and $\tilde{U}(N)$ such that households will have constant intertemporal elasticity of substitution γ, constant Frisch elasticity φ, and χ characterizes the degree to which consumption is more important than labor (high χ implies households cares more about U than $\tilde{U}$).

$$U(C) \equiv \frac{C^{1-\frac{1}{\gamma}} - 1}{1 - \frac{1}{\gamma}} \tag{33}$$

$$\tilde{U}(N) \equiv \frac{1}{\chi} \frac{N^{1+\frac{1}{\varphi}}}{1 + \frac{1}{\varphi}} \tag{34}$$

It is perhaps worth noting γ and φ are the inverses of what many authors use (for example, γ is often used for the relative risk aversion, which is the inverse of the IES). The expressions above then become simpler (dropping time subscripts).

$$C^{\frac{1}{\sigma}} N^{\frac{1}{\varphi}} = \chi \frac{W}{P} \tag{35}$$

$$\frac{\dot{C}}{C} = \gamma(i - \pi - \rho) \tag{36}$$

$$\frac{\dot{N}}{N} = -\varphi(i - \psi - \rho)) \tag{37}$$

3 Firms

Firms seek to maximize the present discounted value of profits. When some degree of nominal rigidity exists, the problem of setting wages is truly a dynamic one, but in the limit without rigidities, the dynamic problem reduces to sequences of static problems at each instant. It will be instructive to first consider this limiting case, then consider the case with rigidity.

3.1 Flexible

Firms take as given the goods price and labor supply schedules of households, and the wages of all other firms. Their production technology is constant returns to scale in labor n, and for simplicity I assume a common productivity Z

$$y(\omega) = Zn(\omega) \tag{38}$$

To simplify notation, I will omit the firm index ω while it is clear that I am focusing on a single firm. The firm problem is

$$\max_{w} \; PZ\left(\frac{w}{W}\right)^{\eta} N - w\left(\frac{w}{W}\right)^{\eta} N \tag{39}$$

$$\tag{40}$$

The first-order condition is

$$0 = \eta \frac{PZ}{w} \left(\frac{w}{W}\right)^{\eta} N - (1+\eta) \left(\frac{w}{W}\right)^{\eta} N \tag{41}$$

Rearranging, we find that the optimal wage is a markdown from marginal revenue product of labor (PZ).

$$w = \frac{\eta}{\eta+1} PZ \tag{42}$$

3.2 Sticky

Firms now face the following quadratic adjustment cost on wages. In this section I let $\psi = \frac{\dot{w}}{w}$ denote wage inflation for an individual firm, which will be its control variable. In equilibrium this distinction will not end up mattering, since all firms will pick the same wage inflation path, but for the purpose of deriving firm optimality, the distinction is crucial, since firms can only control their own wage path, not the aggregate wage path.

$$\Theta(\psi) = \frac{\theta}{2} \psi^2 W N \tag{43}$$

This friction will prevent firms from immediately adjusting their wage following a shock, and will mean they can no longer view their problem as a sequence of decoupled static problems. Instead, their full problem is

$$\mathcal{V}(w_0, t_0) = \max_{\psi} \int_{t_0}^{\infty} e^{-\int_{t_0}^{t} i(s) \mathrm{d}s} \left[P(t) q(n(w(t), t)) - w(t) n(w(t), t) - \frac{\theta}{2} \psi^2 W(t) N(t) \right] \mathrm{d}t \tag{44}$$

$$q(n) = Zn \tag{45}$$

$$n(w, t) = \left(\frac{w}{W(t)} \right)^{\eta} N(t) \tag{46}$$

$$\dot{w} = \psi w \tag{47}$$

$$w(0) = w_0 \tag{48}$$

At first, this may seem overwhelming, but it will turn out to be fairly tame. Without some explanation, the following steps would seem ad hoc, so I will explain the plan of attack before proceeding. First, I find it easier to work with recursive formulations of problems, so I will move from the sequence to the recursive form (this first step is not strictly necessary, though). The second step is to take the first-order condition for the problem, which will guarantee that optimizing behavior is being undertaken. The first-order condition alone will still be in terms of the value function, which is not particularly helpful, so we find the envelope condition in an attempt to remove the value function terms, as we did with the household's problem. It will work (again), and we will be left with an optimal pricing expression purely in terms of aggregates and the current state.

First, to arrive at the recursive formulation, time-differentiate the sequence form of the problem as we did for the consumption-saving problem above, and find the HJB.

$$i\mathcal{V} - \mathcal{V}_t = \max_{\psi} \left[PZn(w) - wn(w) - \frac{\theta}{2} \psi^2 WN + \mathcal{V}_w \psi w \right] \tag{49}$$

The first-order condition relates the marginal cost of higher inflation ($\theta\psi WN$) to the marginal benefit ($\mathcal{V}_w$, the value of a marginally higher wage).

$$\theta\psi WN = \mathcal{V}_w w \tag{50}$$

The $\mathcal{V}_w$ term is not helpful, so we want another expression which relates $\mathcal{V}_w$ with to states and controls. The envelope condition is a good candidate, so we differentiate the HJB (49) with respect to w.

$$i\mathcal{V}_w - \mathcal{V}_{tw} = PZn'(w) - n(w) - wn'(w) + \mathcal{V}_{ww}\psi w + \mathcal{V}_w\psi \tag{51}$$

This is almost useful, but now we have even more $\mathcal{V}$ derivatives. The following trick is perhaps surprising, but often works in these types of problems: time-differentiate the first-order condition to see if that expression will help.

$$\theta\dot{\psi}WN + \theta\psi\dot{W}N + \theta\psi W\dot{N} = \mathcal{V}_{wt}w + \mathcal{V}_{ww}\dot{w}w + \mathcal{V}_w\dot{w} \tag{52}$$

It will! We can combine the time-differentiated first-order condition with the envelope condition to find the optimal wage-setting equations.

$$\theta\frac{W}{w}N\left(i\psi - \frac{\dot{W}}{W}\psi - \frac{\dot{N}}{N} - \dot{\psi}\right) = PZ\eta\left(\frac{w}{W}\right)^{\eta}\frac{N}{w} - (1+\eta)\left(\frac{w}{W}\right)^{\eta}N \tag{53}$$

This equation is not a Phillips curve, though it is tempting to guess it might be. Instead, this summarizes the optimal choice of wage inflation for a firm with current wage w and the other aggregates. Importantly, choosing ψ to satisfy the above expression is optimal regardless of whether $W = w$ or not.

4 Equilibrium

The equilibrium must respect optimality of households and firms, as well goods market clearing, bond market clearing, and labor market clearing. Walras' law allows us to ignore one of the conditions, and typically the bond market is the one we leave out, but we should not forget that, in the background, all households have the opportunity to hold bonds at any time, and the only reason $B = 0$ holds is because the equilibrium interest rate makes households indifferent between holding bonds or not.

4.1 Flexible

In the flexible price equilibrium, prices and wages are not determined in levels, because any scaling of all prices and wages would deliver the same supplies and demands, and thus also be an equilibrium. Relative prices are determined, however, and to find the equilibrium quantity of output we combine the consumption-labor equation of households with the optimal price setting equation of firms, imposing representativeness and all markets clearing. To be concrete,

$$C^{\frac{1}{\gamma}} N^{\frac{1}{\varphi}} = \chi \frac{W}{P} \tag{54}$$

$$W = \frac{\eta}{\eta + 1} PZ \tag{55}$$

$$C = ZN \tag{56}$$

Therefore

$$N^f = \left(\chi \frac{\eta}{\eta + 1} \right)^{\frac{\gamma\varphi}{\gamma+\varphi}} Z^{\frac{(\gamma-1)\varphi}{\gamma+\varphi}} \tag{57}$$

$$C^f = \left(\chi \frac{\eta}{\eta + 1} \right)^{\frac{\gamma\varphi}{\gamma+\varphi}} Z^{\frac{\gamma(1+\varphi)}{\gamma+\varphi}} \tag{58}$$

4.2 Sticky

We now manipulate the optimal pricing expression, imposing equilibrium conditions, to get a Phillips curve. First, impose representativeness ($w = W$)

$$\theta N \left(i\psi - \psi^2 - \frac{\dot{N}}{N} - \dot{\psi} \right) = \frac{P}{W} Z\eta N - (1 + \eta)N \tag{59}$$

This expression can be rearranged to arrive at one form of the New Keynesian Wage Phillips Curve.

$$\dot{\psi} = \psi \left(i - \psi - \frac{\dot{N}}{N} \right) + \frac{\eta + 1}{\theta} \left[1 - \frac{\eta}{\eta + 1} \frac{P}{W} Z \right] \tag{60}$$

A little more work will allow the equation to be expressed purely in terms of inflation and output paths, however. To eliminate the price-wage ratio, use the consumption-labor equation (35) along with market clearing ($C = ZN$).

$$\dot{\psi} = \psi \left(i - \psi - \frac{\dot{N}}{N} \right) + \frac{\eta + 1}{\theta} \left[1 - \frac{\eta}{\eta + 1} \chi Z^{\frac{-\gamma-1}{\gamma}} N^{-\frac{\gamma+\varphi}{\gamma\varphi}} \right] \tag{61}$$

To eliminate the labor time derivative, we can plug in the Euler equation.

$$\dot{\psi} = \psi \left((1 + \varphi)(i - \psi) - \varphi\rho \right) + \frac{\eta + 1}{\theta} \left[1 - \frac{\eta}{\eta + 1} \chi Z^{\frac{\gamma-1}{\gamma}} N^{-\frac{\gamma+\varphi}{\gamma\varphi}} \right] \tag{62}$$

The last piece of the equilibrium is the determination of the nominal interest rate. For simplicity I assume a Taylor rule, where $\phi > 1$, and $\bar{i} \equiv \rho$.

$$i = \bar{i} + \phi\pi \tag{63}$$

The full system is a two-dimensional boundary value problem, which may be summarized below. I now employ lowercase letters for logs (earlier they stood for individual firm variables).

$$\dot{n} = -\varphi(i - \psi - \rho) \tag{64}$$
$$\dot{\psi} = \psi\left((1+\varphi)(i-\psi) - \varphi\rho\right) + \frac{\eta+1}{\theta}\left[1 - \frac{\eta}{\eta+1}\chi \exp\left(\frac{\gamma-1}{\gamma}z - \frac{\gamma+\varphi}{\gamma\varphi}n\right)\right] \tag{65}$$
$$i = \bar{i} + \phi\pi \tag{66}$$
$$\psi - \pi = \frac{1}{\gamma}\dot{c} + \frac{1}{\varphi}\dot{n} = \frac{1}{\gamma}\dot{z} + \frac{\gamma+\varphi}{\gamma\varphi}\dot{n} \tag{67}$$
$$0 = \lim_{T\to\infty} n(T) \tag{68}$$
$$0 = \lim_{T\to\infty} \psi(T) \tag{69}$$
$$\tag{70}$$

4.3 Gaps

The system above works fine, but sometimes we prefer to express it in not in terms of levels, but gaps from the flexible price equilibrium. I will call $X \equiv \frac{N}{N^f}$ the gap between the flexible and realized labor levels. It is also the output gap.

$$X = \frac{N}{N^f} = \frac{ZN}{ZN^f} = \frac{C}{C^f} \tag{71}$$

Let's start with the Euler equation.

$$\dot{x} = \dot{n} - \dot{n^f} \tag{72}$$
$$= \dot{c} - \dot{c^f} \tag{73}$$
$$= \gamma(i - \pi - \rho) - \dot{c^f} \tag{74}$$

The flexible wage path of consumption is easy to calculate[3], and we have

$$\dot{x} = \gamma(i - \pi - \rho) - \frac{\gamma(1+\varphi)}{\gamma\varphi}\dot{z} \tag{75}$$

In the Phillips curve, it is perhaps slightly easier to step back from the log form and note the following.

$$N^f = \left(\frac{\eta}{\eta+1}\chi\right)^{\frac{\gamma\varphi}{\gamma+\varphi}} Z^{\frac{(\gamma-1)\varphi}{\gamma+\varphi}} \tag{76}$$
$$\frac{\eta}{\eta+1}\chi Z^{\frac{\gamma-1}{\gamma}} (XN^f)^{-\frac{\gamma+\varphi}{\gamma\varphi}} = X^{-\frac{\gamma+\varphi}{\gamma\varphi}} \tag{77}$$

Thus the final term in the Phillips curve simplifies, and the full system in terms of inflation and gaps is

[3] Recall the flexible outcome in equation (58). Log the equation and differentiate it to find $\dot{c}^f = \frac{\gamma(1+\varphi)}{\gamma\varphi}\dot{z}$.

$$\dot{x} = \gamma(i - \pi - \rho) - \frac{\gamma(1+\varphi)}{\gamma\varphi}\dot{z} \tag{78}$$

$$\dot{\psi} = \psi\left((1+\varphi)(i-\psi) - \varphi\rho\right) + \frac{\eta+1}{\theta}\left[1 - \exp\left(-\frac{\gamma+\varphi}{\gamma\varphi}x\right)\right] \tag{79}$$

$$i = \bar{i} + \phi\pi \tag{80}$$

$$\psi - \pi = \frac{\gamma+\varphi}{\gamma\varphi}\dot{x} + \dot{z} \tag{81}$$

$$0 = \lim_{T\to\infty} x(T) \tag{82}$$

$$0 = \lim_{T\to\infty} \psi(T) \tag{83}$$

$$\tag{84}$$

5 Aside on comparing to the standard Price NKPC in the special $\gamma = 1$ case

It is worth briefly clarifying a detail that might be bothering you, if you have worked with these models before: why is the Phillips curve (79) not super simple and linear? First, most treatments of NK models perform a linearization at some step. Second, there is a bit of a special secret in the typical NK model with pricing frictions, in that making the intertemporal elasticity of substitution unity kills a few terms. To clarify, the above steps can be followed almost identically in a world with sticky prices to arrive at the following New Keynesian Price Phillips Curve[4]

$$\dot{\pi} = \pi((1-\gamma)(i-\pi) + \gamma\rho) - \frac{\epsilon-1}{\theta}\left[1 - \exp(-\frac{\gamma+\varphi}{\gamma\varphi}x)\right] \tag{85}$$

Imposing $\gamma = 1$ simplifies the first term dramatically, and the approximation (linearization) $1 - \exp(-ax) \approx ax$ leads to the familiar linear form, e.g. the one used by Werning.

$$\dot{\pi} = \rho\pi - \kappa x \tag{86}$$

$$\kappa = \frac{\epsilon-1}{\theta}\frac{1+\varphi}{\varphi} \tag{87}$$

The linearization step can also be completed above, I just did not do so because it does not really make analysis nor numerics any easier. There is no φ analogue for the $\gamma = 1$ trick to eliminate the first terms, since setting $\varphi = -1$ would mean subtracting $\ln N$, and this would mean the labor costs are no longer convex, so the household problem is potentially no longer convex.

6 Computation

An equilibrium system is great, because we are mostly done with the math, and have a framework for thinking about the economics, but since the system is not generally solvable in closed form,

[4] To get here, remove firm power in the labor market ($\eta \to \infty$), add firm power in the product market (finite ϵ), make wages flexible, make prices sticky, and rework through the firm problem and equilibrium conditions. This is an ever so slightly generalized version of Ben Moll's lecture notes.

we need to think about how to tell a computer to solve the problem. As a bonus, the numerical procedure will lend a bit of tatonnement insight into the problem. The type of shocks we are interested in are transitory shocks, here meaning that they affect the paths of variables in the short-run, but in the long-run the economy returns to steady-state.

First, we discretize the problem into nT time steps $1, \ldots, nT$. At each time step we need to check that all endogenous variables satisfy their discretized analogue of the true equilibrium conditions. As a first pass, we can think about guessing paths for wage inflation, price inflation, labor, goods, and the nominal interest rate, then checking all the equilibrium conditions, and searching until all condition are met at all points in time. Besides the fact that this is unwieldy, it is unnecessary, since instead we can guess a much smaller dimensional path and use the equilibrium conditions to determine other endogenous outcomes. The first few reductions are the most obvious:

- The nominal rate is given by goods inflation, so no need to guess i.
- Goods market clearing means $c = z + n$, so no need to guess c.
- Given wage inflation and labor and goods paths, the time-differentiated consumption-labor equation gives goods inflation, so no need to guess π.

Now we are down to wage inflation and the labor path, already a great reduction, but can we do even better? Yes! The Euler equation gives labor as a function of inflation rates, so all we need to guess is a wage inflation path.

What about the boundary conditions? They give us the nT time conditions, so we are left with only $nT - 1$ unknowns (the wage inflation rates), which have to be solved by a computer. Here is a sketch of an algorithm that will do this for us:

(i) Solve the the steady-state equilibrium (it is the flexible equilibrium). This will give us the boundary condition we need for labor.

(ii) Guess a path for wage inflation

(iii) Find the implied nominal interest rate, goods inflation rate, labor path, and consumption path from the equilibrium conditions. To do so, start at the end of time, where we know the boundary condition for n, then solve backwards.

(iv) Using the implied labor path, check if the wage Phillips curve is satisfied for all time. If not, update the wage inflation path and return to (iii).

Here are some bits of economic intuition we can take away from the computation:

- We find the household labor (and consumption) choices by backwards iteration, and this is precisely what forward-looking households do! They take all future prices as given and their optimality conditions give their demands over all time.
- When the Phillips curve is not satisfied, we have found a wage path that satisfies household optimality and all markets clearing, but is not optimal for firms. For example, if we guess an inflation path that is too low, household will work more in earlier times, and firms will realize that by raising wages more quickly, they can extract profit increases that more than offset the higher adjustment costs, so in the next iteration we guess a higher path. This is just as if we were dealing with firms that were learning how to behave over time, but made mistakes early on, so our iterations on the wage path may be thought of as "teaching" the firms their optimality conditions, and we stop when they are optimized (up to machine precision).

7 Results

Now that we have set up the model and defined the equilibrium system, it is time to see what it can actually do! I discuss four different shock types, all perfect foresight transitory MIT shocks[5]. Although comparative statics on the dynamic equilibrium are not immediately analytically available, I provide some economic intuition for each set of outcomes. One point that may be helpful to keep in mind as reading through these scenarios is that the flexible outcome deviates from steady-state only in the Figure 1. The simple reason is that the flexible outcome depends only on TFP and parameter values other than the discount rate, so shocks to the interest rate, discount rate, or "cost-push" are fully undone in the flexible case by full and immediate wage adjustment, whereas wage rigidity allows these nominal shocks to alter time paths of real outcomes.

The shock in Figure 1 is a temporary reduction in productivity z. Firms see their productivity drop, and see the goods price drop, and understand its future inflationary path, so in response deflate wages to mitigate profit losses along the transition. The central bank sees goods prices inflating and raises the nominal interest rate accordingly. Households see the real interest rate rise, and want to save more today, which they try to achieve by both consuming less and working more, so as to satisfy their consumption-labor intratemporal condition. The friction in wages will mean that the real wage is higher than it would be on the flexible path, and accordingly labor increases more than it would on the flexible path. They key is that the shock only directly affects producers, but for equilibrium to hold, all other prices must move as described.

The following experiment is a bit surprising. Suppose the central bank, beyond its normal Taylor rule, suddenly considers an additional positive shock $\tilde{i}$ to the interest rate ((63) becomes $i = \bar{i} + \phi\pi + \tilde{i}$), which again decays exponentially, depicted in Figure 2. Firms see prices jump up and understand they will deflate moving forward, so inflate their wages to increase their output and take advantage of the high price in the short run. The price deflation is strong enough to *change the sign* of the nominal rate change via the ϕ term in the Taylor rule, and the nominal rate falls on impact, but not more than inflation (due to the rule shock), so the real rate rises. The high real rate causes households to want to save more and consume less, but the fall in the real wage also causes them to work less. The effect of the real wage decline is larger than the intertemporal substitution effect, so the output gap falls on impact. The shock affects both households and producers directly by changing the value of saving (for households) and discounting (for producers), and indirectly affects households through changing the equilibrium real wage.

Now suppose households temporarily become more impatient, meaning that ρ is temporarily increased, in Figure 3. Households want to consume more today, particularly because goods prices decreased on impact, and this increase in goods demand causes inflation and interest rates to jump up. Additionally, since real wages jumped up, households work more. Firms also care more about the present with the higher interest rates, and as usual adjust wages, in this case deflating them to mitigate losses along the transition. Since wages cannot move as quickly as in the flexible case, the real wage and increased consumption demand causes the output gap to be positive. The discount rate shock only directly affects households, but for the goods market to clear the real wage must rise enough to generate sufficient labor supply to produce goods demanded.

Lastly, consider a "cost-push" shock in Figure 4, which is just an extra term on the end of the wage Phillips curve (right side of (79) gets a $+u$ on the end, where u is the shock). Essentially, this shock means that, for some not modelled reason, all firms face an additional force which causes them to increase wage inflation faster than they otherwise would. In order for this to be an equilibrium, prices jump down on impact, and deflate more slowly than wages, because a larger

[5] I am happy to share the code used to generate these figures, but have not posted it yet purely because it is not cleaned up yet.

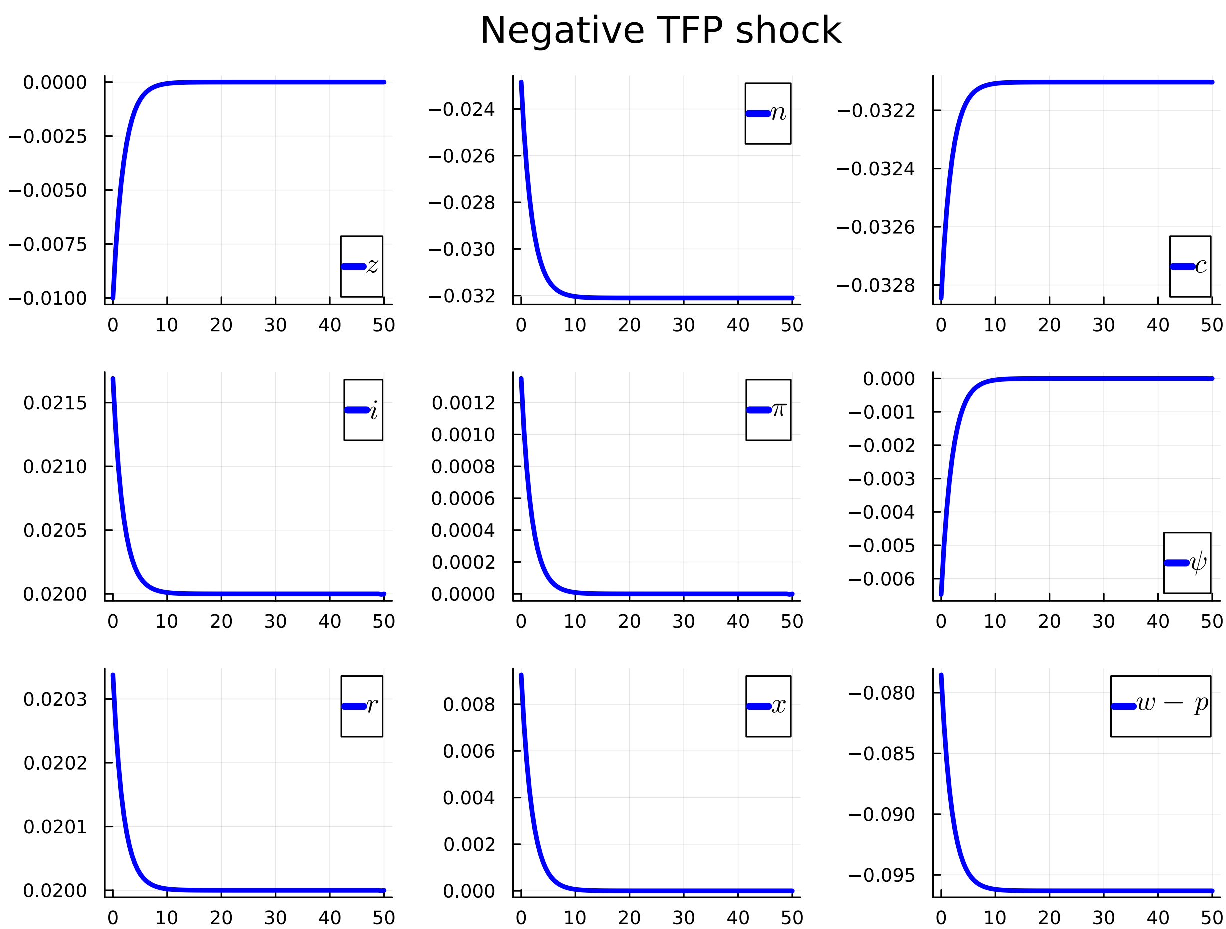

Figure 1

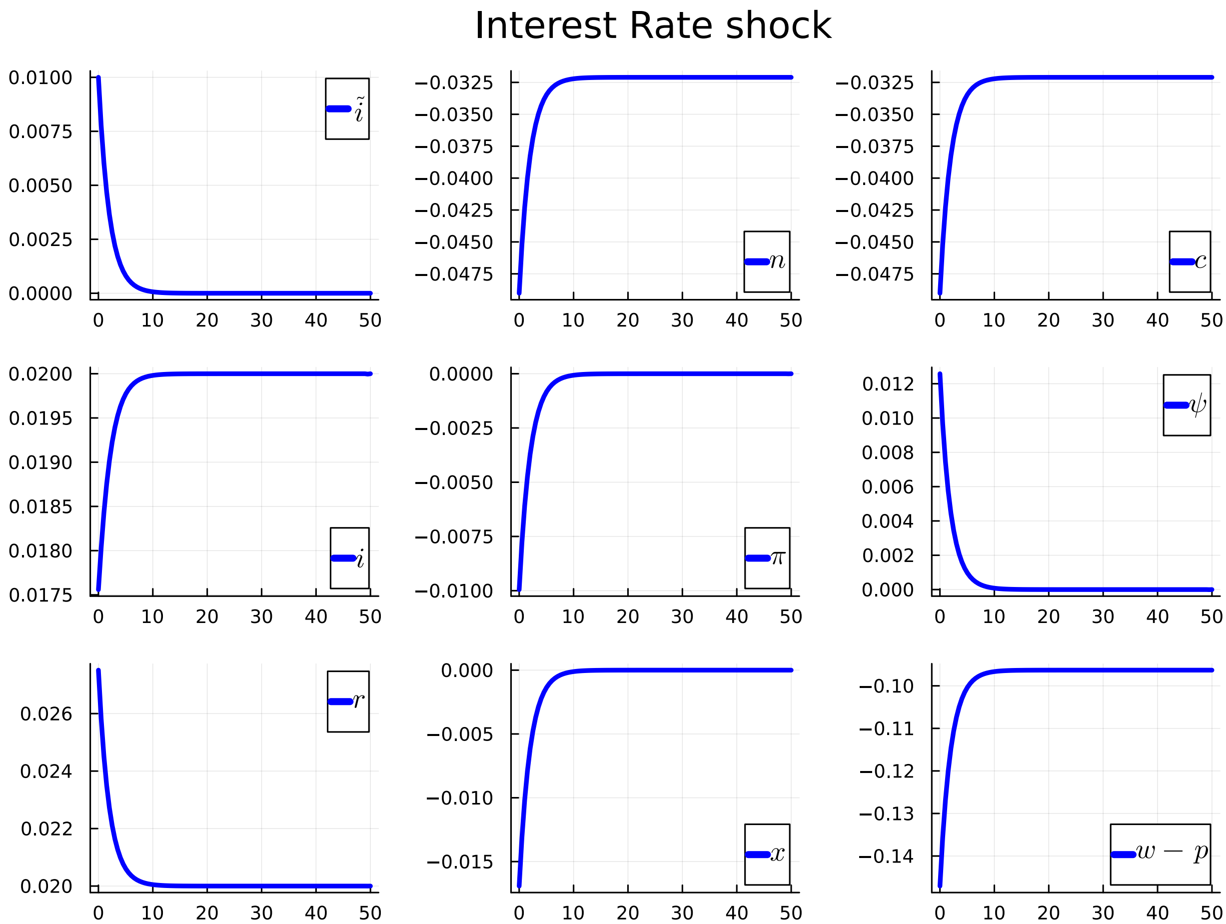

Figure 2

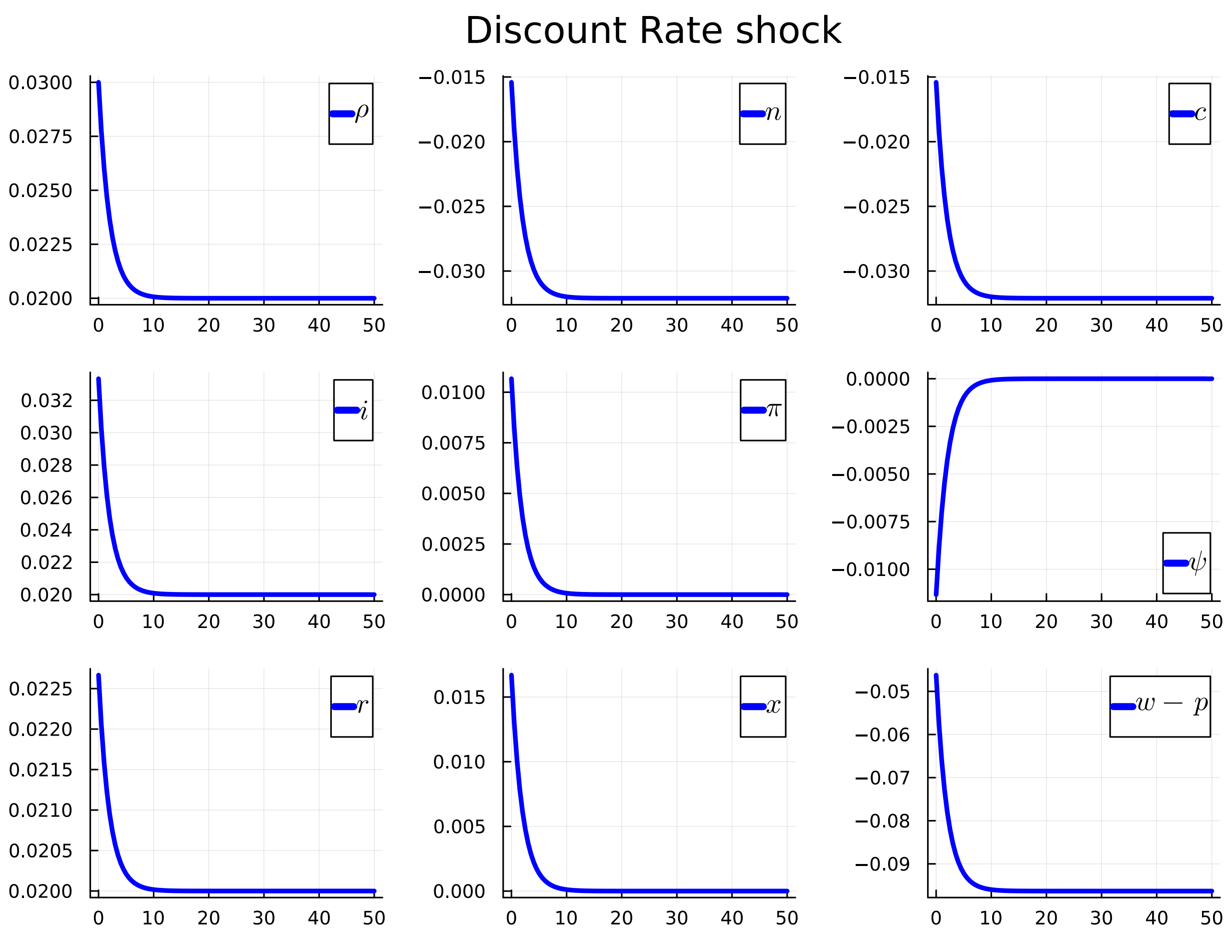
Discount Rate shock
ρ
n
c
i
π
ψ
r
x
$w - p$
0.0300
0.0275
0.0250
0.0225
0.0200
−0.015
−0.020
−0.025
−0.030
0.032
0.030
0.028
0.026
0.024
0.022
0.020
0.0100
0.0075
0.0050
0.0025
0.0000
−0.0025
−0.0050
−0.0075
−0.0100
0.0225
0.0220
0.0215
0.0210
0.0205
0.015
0.010
0.005
0.000
−0.05
−0.06
−0.07
−0.08
−0.09
0
10
20
30
40
50

Figure 3

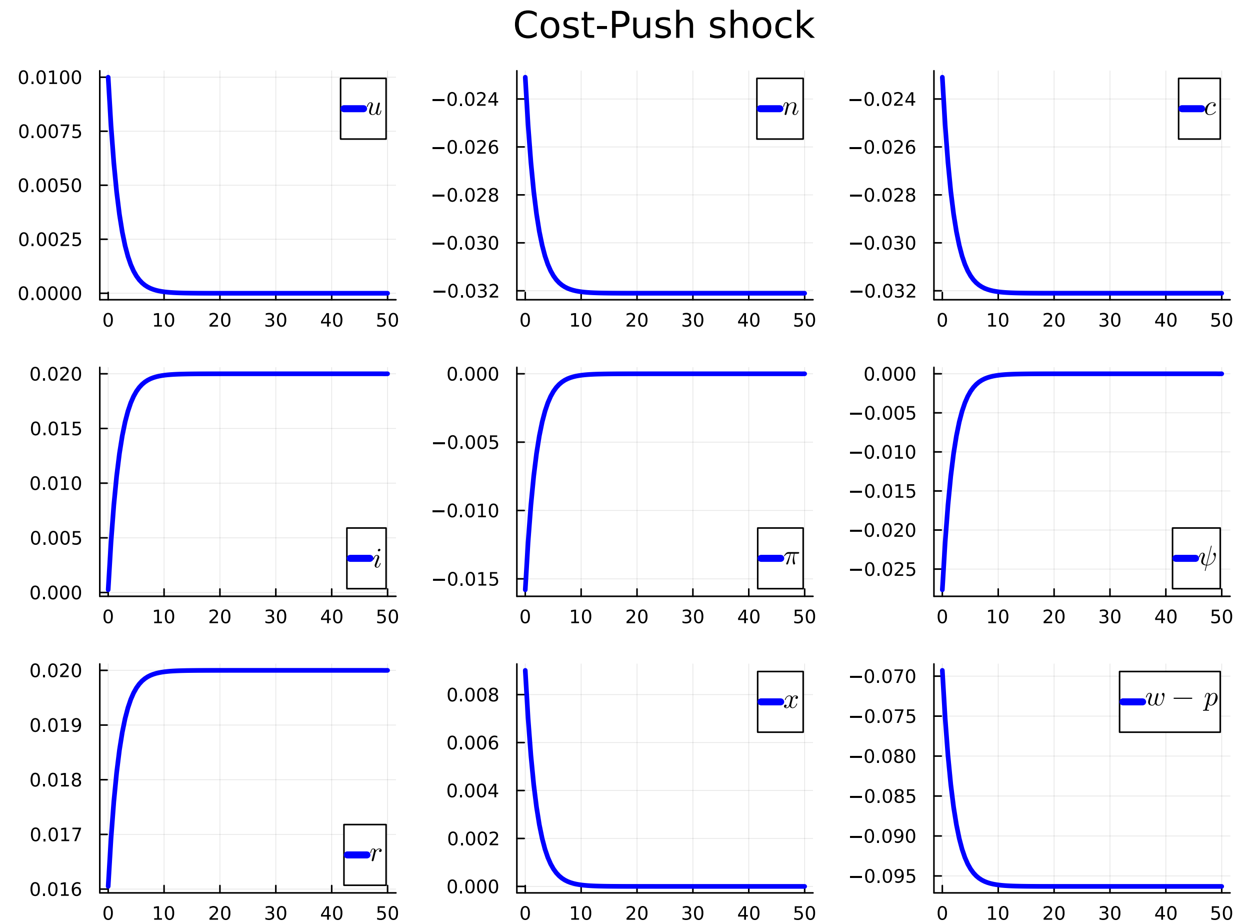

Figure 4

price deflation rate would cause the interest rate to fall too far, inducing too much goods demand. Households do see the interest rates decline, however, and consume more, but the increased real wage causes them to also work more. The wage friction induces the real wage hike, which therefore causes consumption and labor to be above their flexible outcomes. This shock only directly affects producers, but since it affects price paths, and inflation affects the interest rate via monetary policy, households change their labor supply and goods demand accordingly.

8 Closing Thoughts

This NK model passes the basic modelling test of starting with simple assumptions and arriving at interesting (and maybe sometimes correct?) conclusions. We assume agents like to consume, dislike work, consumption/labor smooth using bonds, firms have some monopsonistic power and maximize present discounted value profits, and the central bank raises nominal interest rates in response to inflation. We arrive at conclusions that negative TFP shocks and surprise interest rate hikes generate output recessions, and positive demand shocks and surprise bonus wage inflation (cost-push shock) generate output booms. Are these realistic outcomes? Maybe. Are they a good starting point for trying to understand why nominal rigidities can have real effects? I say yes.

My hope is that, regardless of if you are seeing the NK model for the first time or are a seasoned vet (I put myself much closer to the former than the latter), this note provides some insights into how to derive and solve the model in a comparatively simple way. While the model is now standard in the toolkit of economists thinking about inflation, I have still found many expositions leave my head spinning, or wondering why half the steps were taken (or even if they were legal mathematical steps!). Perhaps the above steps also fall into this category, but if they provide any new illumination, I'll be satisfied.