Welfare Theorems

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1 What are they?

First we need some terminology.

- A **competitive equilibrium** is a set of prices and quantities such that, given prices, agents are optimizing, budgets clear, and feasibility constraints are satisfied. Note that this definition encompasses models with firms, price stickiness, etc., but it must be the case that all agents see the same prices (i.e. no wedges). The theorem below is specific to an endowment trading economy with only consumers.
- A **decentralized equilibrium** or **competitive equilibrium with taxes** is the same as a competitive equilibrium, except that different agents may see different effective prices.
- An allocation (not saying anything about prices) is **Pareto efficient** or **Pareto optimal** if there does not exist another feasible allocation such that all agents are at least as well off, and at least one agent is strictly better off.
- Locally non-satiatied preferences are such that, for any given x, and any given neighborbood $N(x)$ of x, there exists $y \in N(x)$ such that y is preferred to x. Basically, preferences never peak or even plateau in the goods space.

2 FWT

The First Welfare Theorem: Suppose an economy is populated by a set $\mathcal{I}$ of agents, each with endowment e_i of goods, and with locally nonsatiated preferences. Then a competitive equilibrium is Pareto efficient...

Proof: Let $\mathbf{p}$ be the competitive equilibrium prices, and $\mathbf{x}^*$ the competitive equilibrium allocation.

Then for agent i, if bundle x_i is preferred to bundle x_i^*, it must be that $\mathbf{p}x_i > \mathbf{p}x_i^*$. If this were not the case, then $\mathbf{p}x_i \leq \mathbf{p}x_i^* \leq \mathbf{p}e_i$, so x_i is budget feasible and preferred, and agent i is not optimizing, a contradiction. Furthermore, if agent i is indifferent between x_i and x_i^*, it must be that $\mathbf{p}x_i \geq \mathbf{p}x_i^*$, otherwise, by local non-satiation, a budget feasible deviation would exist from x_i which would be preferable to x_i^*. In particular, if $\mathbf{p}x_i < \mathbf{p}x_i^*$, then the ball $B_{\frac{\mathbf{p}(x_i^* - x_i)}{p_{\text{sup}}}}(x_i)$ is feasible, where p_{sup} is the supremum of the prices, and by locally nonsatiated preferences, there is an allocation in this ball which is preferable, thus preferable to x_i^* also.

Now suppose $(\mathbf{p}, \mathbf{x}^*)$ is not Pareto efficient. Then there exists $\mathbf{x}$ in which all agents are at least as well off, and at least one agent is strictly better off. By the argument in the previous paragraph, $\mathbf{p}x_i \geq \mathbf{p}x_i^*$ for all agents, with strictness for at least one. Then we can say

$$\begin{aligned}
\mathbf{p}\sum_{i\in\mathcal{I}} x_i &= \sum_{i\in\mathcal{I}} \mathbf{p}x_i \\
&> \sum_{i\in\mathcal{I}} \mathbf{p}x_i^* \\
&= \sum_{i\in\mathcal{I}} \mathbf{p}e_i \\
&= \mathbf{p}\sum_{i\in\mathcal{I}} e_i
\end{aligned}$$

Since $\mathbf{p}$ is non-negative, we must have at least one element of $\sum_i x_i$ is greater than at least one element of $\sum_i e_i$. But then the allocation is not feasible, a contradiction to the supposed Pareto efficiency. So $\mathbf{x}^*$ is Pareto efficient. □

Now there is something stinky going on here. I never specified the type of set for agents or goods, yet I went ahead and assumed I could make the crucial comparative argument above. In fact, I used a $\sum_i$ when in fact an integral might have been more appropriate, say for a continuum of agents. Thus, without further conditioning the theorem, it is possible the comparative argument above might turn into an $\infty > \infty$ which is not only weird, but actually breaks the theorem! So we amend

First Welfare Theorem (cont.): ... provided that the competitive equilibrium satisfies $\int_{\mathcal{I}} \mathbf{p}x_i < \infty$.

When might this apply? We might have an infinite number of agents as in an overlapping generations model, or finite number of agents, but living forever, thus infinitely many goods.

3 SWT

Second Welfare Theorem: If our economy is sufficiently convex, then a Pareto efficient allocation $\mathbf{x}$ can be supported by a price vector $\mathbf{p}$ such that $(\mathbf{p}, \mathbf{x})$ comprises a competitive equilibrium.

I omit the proof, as at the moment it does not provide any interesting insights.

4 Discussion

As an economist, you should at least one time in your life take a moment to stop and admire the wildness that is a competitive equilibrium (now is good). Mathematically, this is just a fixed point of a system, but it's quite surprising that a bunch of agents, all acting purely in their own self-interest, are able to generate a system which has a fixed point. This is essentially part of the invisible hand argument.

A socially planned allocation is a priori *completely different*, wherein a dictator considers all the inputs (preferences, technology) for a system, then allocates according to some metric, such as Pareto efficiency. It should be truly mind-blowing that letting all the agents just act in their own self-interest would yield the same choice as a social planner seeking Pareto efficiency. But this is what the First Welfare Theorem says can happen! With this perspective, it's not at all surprising that this result may be broken, e.g. with certain social security problems.

Macroeconomists like the welfare theorems because they often give us a way to jump between a social planner problem and a decentralized problem (i.e. competitive equilibrium). For example, finding a competitive equilibrium may potentially be a beast of a task, but perhaps solving the planner problem with given Pareto weights is quite easy, and we know how to map Pareto weights to endowment allocations. Then if the problem satisfies the assumptions needed for the Second Welfare Theorem (which is often the case), we can solve the competitive equilibrium by solving for the social planner problem. To intuitively see why this might work, consider the following: would you rather solve an $I \times n$-dimensional constrained optimization problem, or I number of n-dimensional constrained optimization problems? At the same time. Oh and also to do this you also basically have to smart guess-and-verify, because each agent takes prices as given, so to test if a price combination works, you have to see how all agents optimize, then check if this outcome is feasible.[1]

Another application is to test for if policy intervention could make the agents better off. For example, if the FWT holds, then there is no way to improve upon the competitive equilibrium, if by improve we mean making no one worse off, and making at least someone better off. But when we break FWT, it might be the case that a social planner can actually intervene and improve all welfare. The classic example is social security, where a CE will not deliver what we want, because I need to pay the old while I'm young, then be paid by the young when I'm old, and under certain conditions there is no way to achieve this contract.

[1] Obviously I'm being a bit facetious here, and in fact this exact process is what is often used and state of the art for solving macro models; we've just gotten smarter at the nuts and bolts of the guess-and-verify.